TRaitement des Images pour la Vision Artificielle

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slides in part based on material from Mathieu Aubry, Andrew Zisserman, David Lowe

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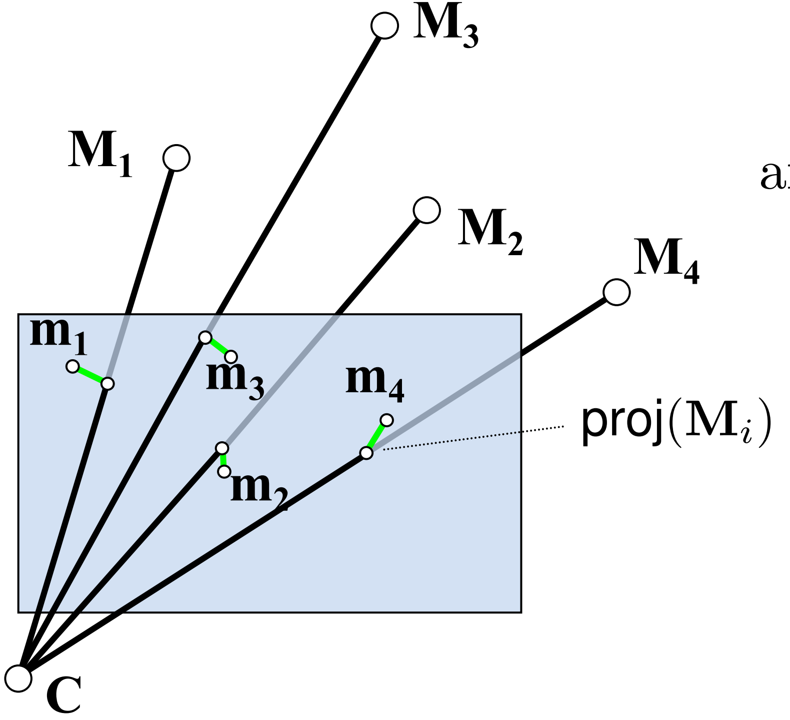
Bayesian Theory for Computer Vision

In the previous lecture



$$\arg\min_{\mathbf{R},\mathbf{T}} \frac{1}{|\mathcal{M}|} \sum_i \| \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}) - \mathbf{m}_i \|^2$$

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In this lecture

$$\arg\max_{\mathbf{x}} \; p(\mathbf{x} \mid \mathbf{z})$$

$\mathbf{x}$: state [in our case, for example $\mathbf{R}$, $\mathbf{T}$]
$\mathbf{z}$: observations [in our case, for example feature points $\mathbf{m}_i$]

$p(\mathbf{x} \mid \mathbf{z})$: probability density function of $\mathbf{x}$ given $\mathbf{z}$

probability density function

$p(\mathbf{x})$: probability density function (pdf) of $\mathbf{x}$

Intuitively:

- $p(\mathbf{x})$ ~ how likely the random variable for $\mathbf{x}$ is close to $\mathbf{x}$
- $p(\mathbf{x}_1) > p(\mathbf{x}_2)$: the correct value is more likely to be $\mathbf{x}_1$ than $\mathbf{x}_2$

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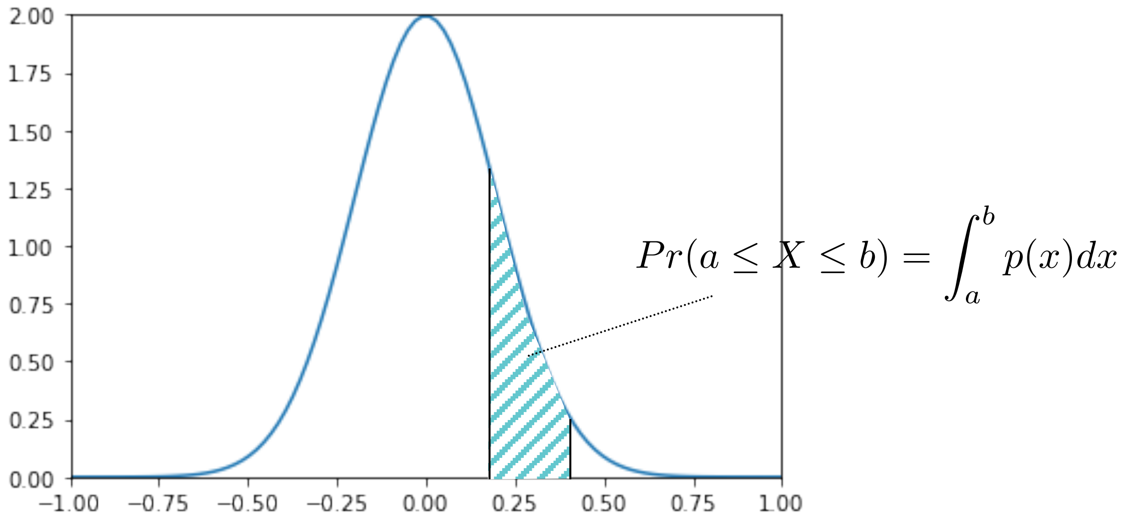
probability density function (2)

For a univariate random variable X:

$$Pr(a \leq X \leq b) = \int_a^b p(x)dx$$

$$p(x) = N(x; 0, 0.2)$$



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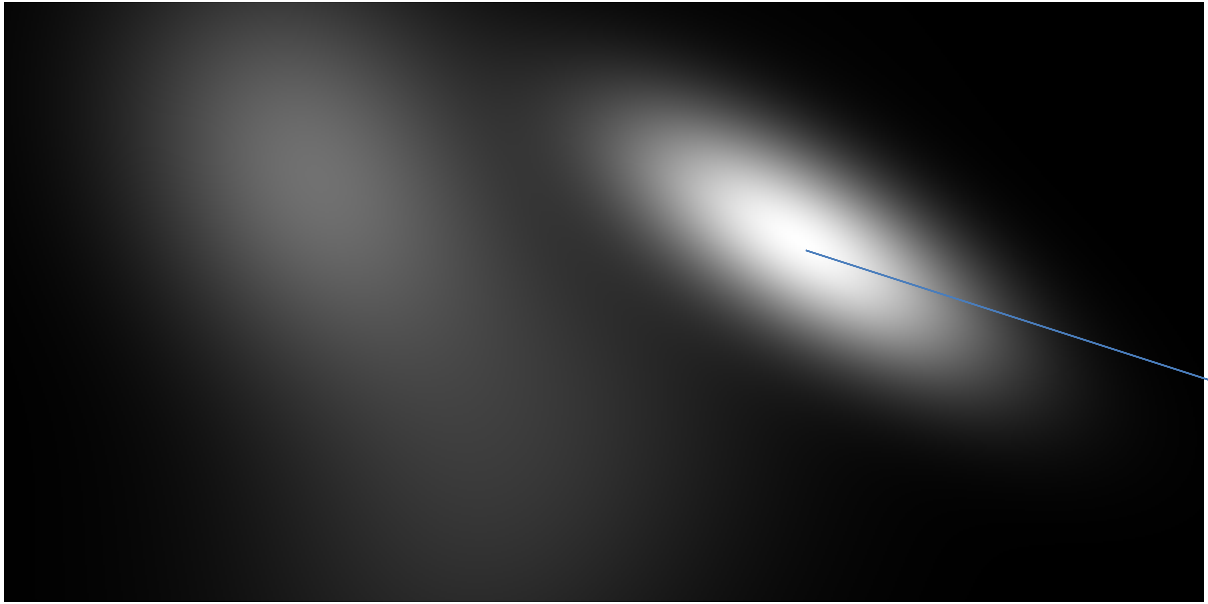
probability density function (2)

For a 2D random variable:



$$\arg\max_{\mathbf{x}} \; p(\mathbf{x} \mid \mathbf{z})$$

$$p(\mathbf{x} \mid \mathbf{z})$$

Bayes' theorem

$$\arg\max_{\mathbf{x}} \; p(\mathbf{x} \mid \mathbf{z})$$

$\mathbf{x}$: state [in our case, for example $\mathbf{R}$, $\mathbf{T}$]
$\mathbf{z}$: observations [in our case, for example feature points $\mathbf{m}_i$]

$p(\mathbf{x} \mid \mathbf{z})$: probability density function of $\mathbf{x}$ given $\mathbf{z}$

$$\underbrace{p(\mathbf{x} \mid \mathbf{z})}_{\text{posterior}} = \frac{1}{p(\mathbf{z})} \underbrace{p(\mathbf{z} \mid \mathbf{x})}_{\text{likelihood}} \underbrace{p(\mathbf{x})}_{\text{prior}}$$

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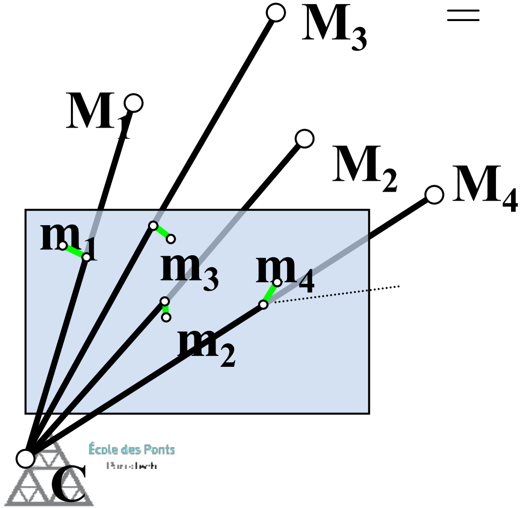
Likelihood - example

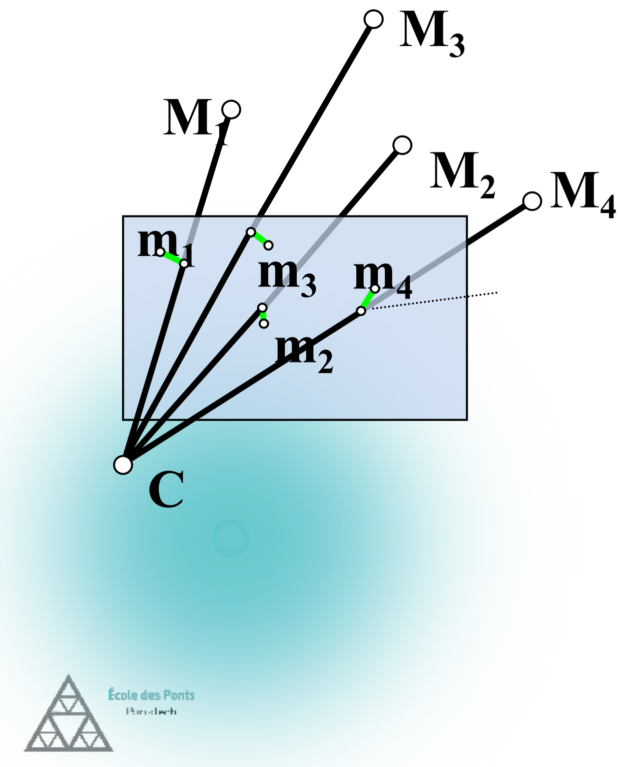
$$
\begin{aligned}
p(\mathbf{z} \mid \mathbf{x}) &= p(\mathbf{z}_1, .., \mathbf{z}_n \mid \mathbf{x}) \\
&= \prod_i p(\mathbf{z}_i \mid \mathbf{x}) \\
&= \prod_i N(\mathbf{m}_i; \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma)
\end{aligned}
$$

if we assume
the observations $\mathbf{z}_i$
are independent given $\mathbf{x}$

if we assume the noise
on the observations
is Gaussian, with covariance Σ



Prior



If we have no idea what the state can be:

$$p(\mathbf{x}) = \mu$$

If we believe the camera center is close to Point (0,0,0):

$$p(\mathbf{x}) = N(\mathbf{C}(\mathbf{x}); \mathbf{0}, \Sigma_{\mathbf{x}})$$

etc.

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Link (1)

$$
\begin{aligned}
& \arg\max_{\mathbf{x}} \; p(\mathbf{x} \mid \mathbf{z}) \\
= \quad & \arg\max_{\mathbf{x}} \; \frac{1}{p(\mathbf{z})} p(\mathbf{z} \mid \mathbf{x}) p(\mathbf{x}) \\
= \quad & \arg\max_{\mathbf{R},\mathbf{T}} \; \prod_i N(\mathbf{m}_i; \mathsf{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma) \\
= \quad & \arg\min_{\mathbf{R},\mathbf{T}} \; -\log \prod_i N(\mathbf{m}_i; \mathsf{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma) \\
= \quad & \arg\min_{\mathbf{R},\mathbf{T}} \; -\sum_i \log N(\mathbf{m}_i; \mathsf{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma) \\
= \quad & \arg\min_{\mathbf{R},\mathbf{T}} \; \sum_i \|\mathbf{m}_i - \mathsf{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T})\|^2
\end{aligned}
$$

Link (2)

$$
\begin{aligned}
& \arg\max_{\mathbf{x}}\ p(\mathbf{x} \mid \mathbf{z}) \\
= \quad & \arg\max_{\mathbf{x}}\ \frac{1}{p(\mathbf{z})} p(\mathbf{z} \mid \mathbf{x}) p(\mathbf{x}) \\
= \quad & \arg\max_{\mathbf{R},\mathbf{T}}\ \prod_i N(\mathbf{m}_i; \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma)\ N(\mathbf{C}(\mathbf{R}, \mathbf{T}); \mathbf{0}, \Sigma_{\mathbf{x}}) \\
= \quad & \arg\min_{\mathbf{R},\mathbf{T}}\ -\sum_i \log N(\mathbf{m}_i; \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma) + \log N(\mathbf{C}(\mathbf{R}, \mathbf{T}); \mathbf{0}, \Sigma_{\mathbf{x}}) \\
= \quad & \arg\min_{\mathbf{R},\mathbf{T}}\ \sum_i \|\mathbf{m}_i - \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T})\|^2 + \lambda \|\mathbf{C}(\mathbf{R}, \mathbf{T})\|^2
\end{aligned}
$$

log-likelihood regularization term

Robust estimators

If we assume the noise on the observations is Gaussian:

$$p(\mathbf{z}_i \mid \mathbf{x}) = N(\mathbf{m}_i; \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma)$$

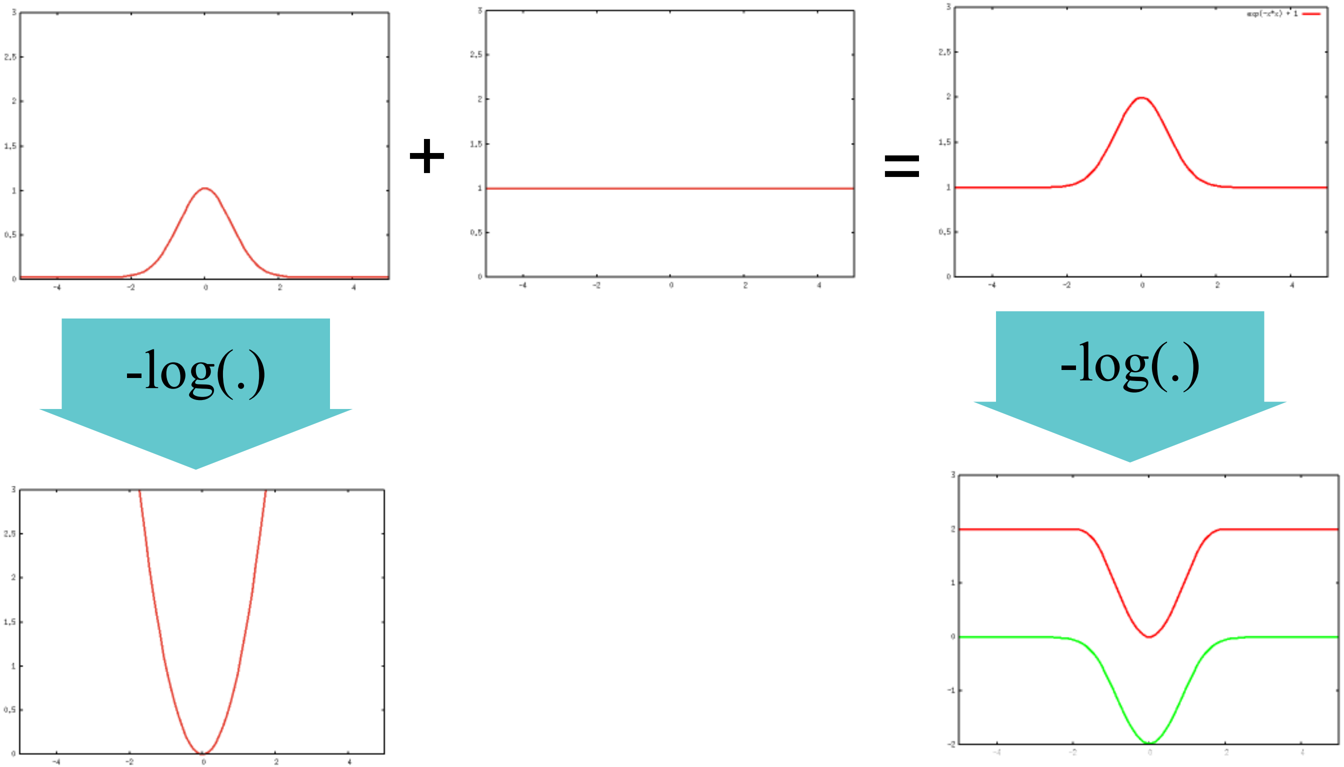
If we consider that $\mathbf{m}_i$ can be either an inlier with Gaussian noise or an outlier, we can take:

$$p(\mathbf{z}_i \mid \mathbf{x}) \sim N(\mathbf{m}_i; \text{proj}(\mathbf{M}_i; \mathbf{R}, \mathbf{T}), \Sigma) + \mu$$

Normal distribution (inliers)

Uniform distribution (outliers)

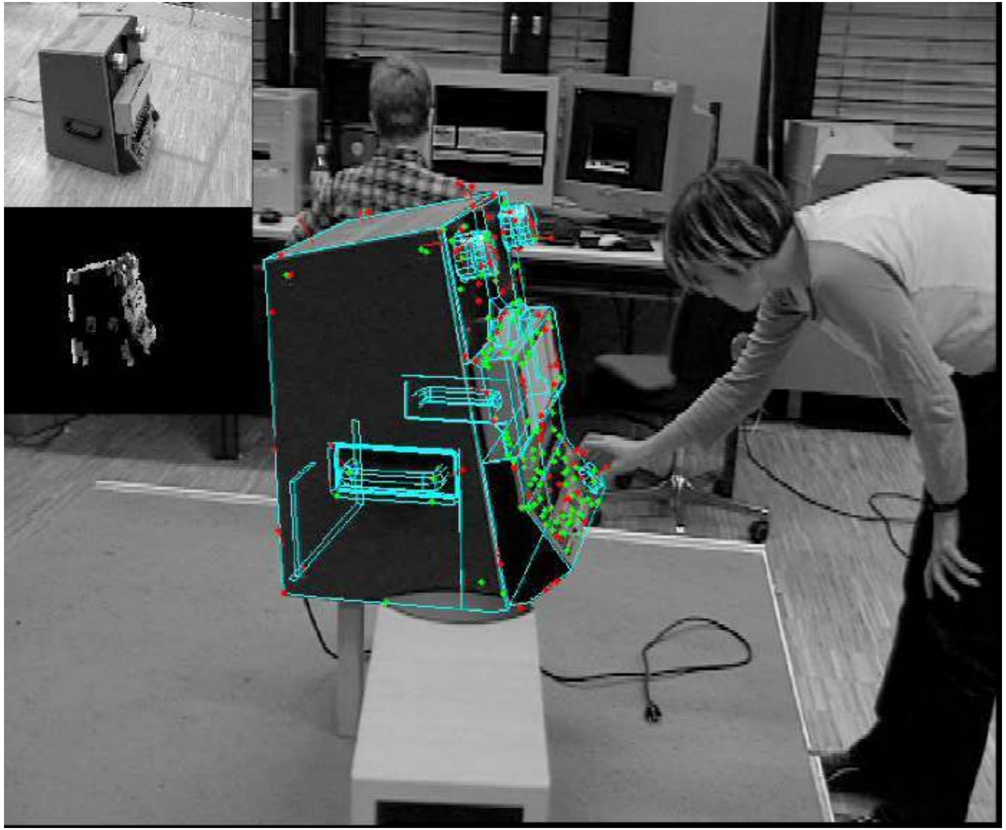
Mixture



Least-squares

Tukey estimator

Tracking over Time



Tracking over Time

$\mathbf{x}_t$: State at time t

$\mathbf{X}_t = \mathbf{x}_t\,,\ \mathbf{x}_{t\text{-}1}\,, \ldots\ \mathbf{x}_0$: States from time 0 to time t

$\mathbf{z}_t$: Observations performed at time t.

$\mathbf{Z}_t = \mathbf{z}_t\,,\ \mathbf{z}_{t\text{-}1}\,, \ldots\ \mathbf{z}_0$: Observations up to time t

How can we estimate the density $p(\mathbf{x}_t \mid \mathbf{Z}_t)$?

ie What is the density on the object state at time t given the images captured up to time t?

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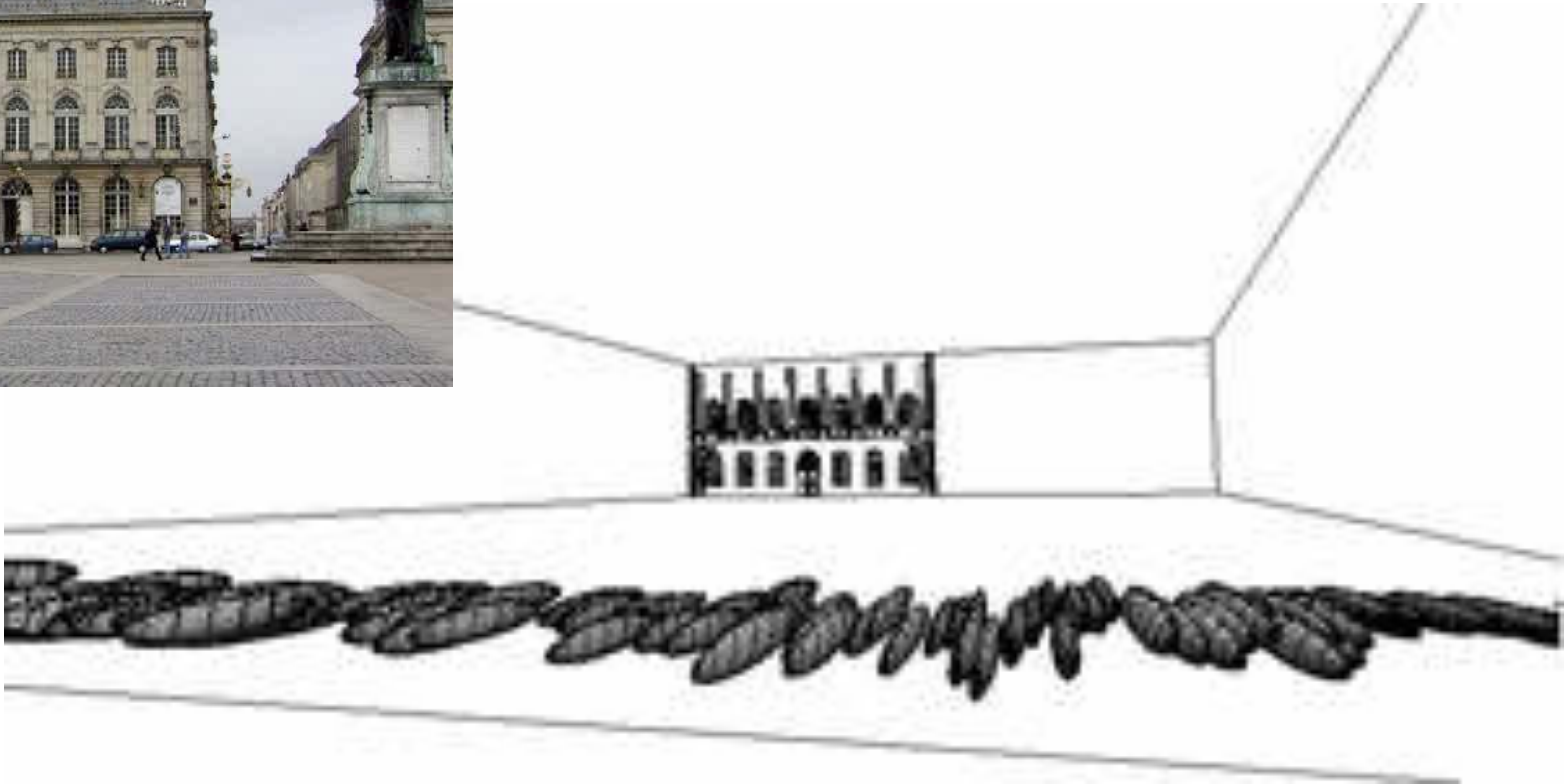
Example



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Tracking over Time

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How can we estimate the density $p(\mathbf{x}_t \mid \mathbf{Z}_t)$?

ie What is the density on the object state at time t given the images captured up to time t?

We will see that:

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \sim \underbrace{p(\mathbf{z}_t \mid \mathbf{x}_t)}_{\text{likelihood / Observation model}} \int_{\mathbf{x}_{t-1}} \underbrace{p(\mathbf{x}_t \mid \mathbf{x}_{t-1})}_{\text{Motion model}} \underbrace{p(\mathbf{x}_{t-1} \mid \mathbf{Z}_{t-1})}_{\text{Density estimated at time } t\text{-}1}$$

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$p(\mathbf{X}_t, \mathbf{Z}_t) = p(\mathbf{x}_t, \mathbf{z}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$ [definition]

$= p(\mathbf{x}_t, \mathbf{z}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$ [conditional probability: $p(A, B) = p(A \mid B)\, p(B)$]

$= p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$ [conditional probability]

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We just saw:

$p(\mathbf{X}_t, \mathbf{Z}_t) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$

We also have:

$p(\mathbf{X}_t, \mathbf{Z}_t) = p(\mathbf{X}_t \mid \mathbf{Z}_t)\, p(\mathbf{Z}_t)$

so

$p(\mathbf{X}_t \mid \mathbf{Z}_t)\, p(\mathbf{Z}_t) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t)\, p(\mathbf{Z}_t) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})$$

We also have:

$$p(\mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1}) = p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})\, p(\mathbf{Z}_{t\text{-}1})$$

so

$$p(\mathbf{X}_t \mid \mathbf{Z}_t)\, p(\mathbf{Z}_t) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})\, p(\mathbf{Z}_{t\text{-}1})$$

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t)\, p(\mathbf{Z}_t) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})\, p(\mathbf{Z}_{t\text{-}1})$$

$p(\mathbf{Z}_t)$ is constant with respect to $\mathbf{x}_t$, so

$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

We want to get to:

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t) \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}) p(\mathbf{x}_{t-1} \mid \mathbf{Z}_{t-1})$$

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

We assume the observations $\mathbf{z}_0$, $\mathbf{z}_1$, ..., $\mathbf{z}_t$ are independent both (1) mutually and (2) with respect to the dynamical process:

$$p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1}) = p(\mathbf{z}_t \mid \mathbf{x}_t)$$

We now have:

$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

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Tracking over Time

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

We want to get to:

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t) \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}) p(\mathbf{x}_{t-1} \mid \mathbf{Z}_{t-1})$$

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$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{X}_{t\text{-}1}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

We can make $\mathbf{x}_{t\text{-}1}$ appear:

$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1}, \mathbf{X}_{t\text{-}2}, \mathbf{Z}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

We assume the states form a Markov chain: $p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1}, \mathbf{X}_{t\text{-}2}, \mathbf{Z}_{t\text{-}1}) = p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})$

So:

$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

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$\mathbf{Z}_t = \mathbf{z}_t , \mathbf{z}_{t\text{-}1} , \dots \mathbf{z}_0$: Observations up to time t

We had:

$$p(\mathbf{X}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})\, p(\mathbf{X}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

Making $\mathbf{X}_{t\text{-}2}$ appear:

$$p(\mathbf{x}_t , \mathbf{x}_{t\text{-}1} , \mathbf{X}_{t\text{-}2} \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})\, p(\mathbf{x}_{t\text{-}1} , \mathbf{X}_{t\text{-}2} \mid \mathbf{Z}_{t\text{-}1})$$

$$\int_B p(A , B) = p(A)$$

Integrating over $\mathbf{X}_{t\text{-}2}$:

$$p(\mathbf{x}_t , \mathbf{x}_{t\text{-}1} \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})\, p(\mathbf{x}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

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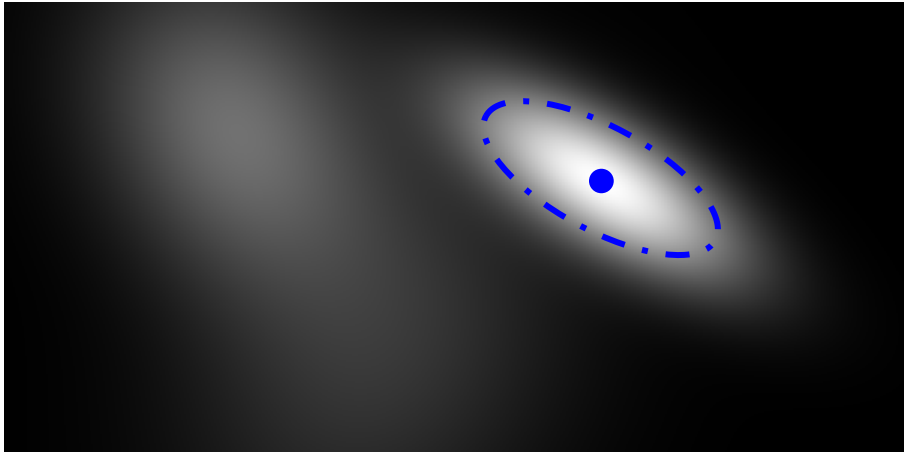
$$p(\mathbf{x}_t , \mathbf{x}_{t\text{-}1} \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{x}_{t\text{-}1})\, p(\mathbf{x}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1})$$

Integrating over $\mathbf{x}_{t\text{-}1}$:

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t) \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}) p(\mathbf{x}_{t-1} \mid \mathbf{Z}_{t-1})$$

Kalman Filter

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) = N(\mathbf{x}_t; \mu_t, \Sigma_t)$$



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Time update or Prediction

The states $\mathbf{x}_t$ are assumed to evolve according to a dynamics model of the form:

$$\mathbf{x}_t = \mathbf{A}\,\mathbf{x}_{t\text{-}1} + \mathbf{w}_t$$

where:

- matrix $\mathbf{A}$ is a state transition matrix;
- vector $\mathbf{w}_t$ is the process noise.

State transition matrix $\mathbf{A}$

Example:

If we assume the motion model is a constant velocity model, with no rotation:

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{T}_t \\ \dot{\mathbf{T}}_t \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{T}_{t+1} \\ \dot{\mathbf{T}}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{T}_t \\ \dot{\mathbf{T}}_t \end{bmatrix} + \mathbf{w}_t$$

The measurement update or Correction

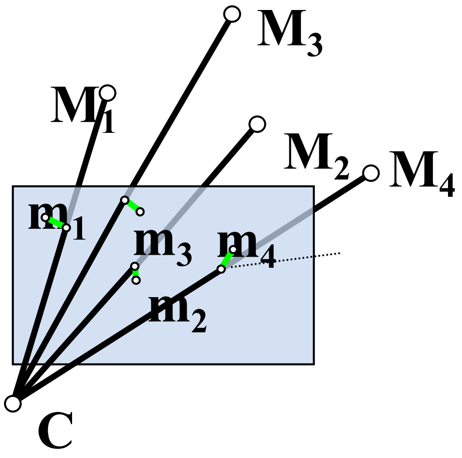
Linear case:

The measurements $\mathbf{z}_t$ are assumed to be related to the state $\mathbf{x}_t$ by a linear measurement model:

$$\mathbf{z}_t = \mathbf{C}\,\mathbf{x}_t + \mathbf{v}_t$$

where

the vector $\mathbf{v}_t$ represents the measurement noise.



This is not linear but it is always possible to linearize it.

Kalman filter

$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \sim p(\mathbf{z}_t \mid \mathbf{x}_t) \int_{\mathbf{x}_{t-1}} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}) p(\mathbf{x}_{t-1} \mid \mathbf{Z}_{t-1})$$

Under the assumptions that: $\mathbf{z}_t = \mathbf{C}\ \mathbf{x}_t + \mathbf{v}_t$ and $\mathbf{x}_t = \mathbf{A}\ \mathbf{x}_{t\text{-}1} + \mathbf{w}_t$

If we know $p(\mathbf{x}_{t\text{-}1} \mid \mathbf{Z}_{t\text{-}1}) = N(\mathbf{T}_{t\text{-}1}; \mu_{t\text{-}1}, \Sigma_{t\text{-}1})$, can we compute $p(\mathbf{x}_t \mid \mathbf{Z}_t)$?

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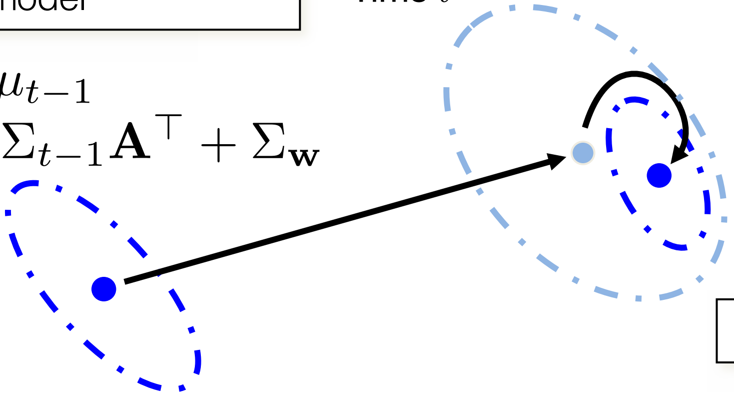
The Kalman filter

1. Time update or Prediction
→ Uses a motion model

$$\begin{cases} \mu_t^- = \mathbf{A}\mu_{t-1} \\ \Sigma_t^- = \mathbf{A}\Sigma_{t-1}\mathbf{A}^\top + \Sigma_\mathbf{w} \end{cases}$$

Time t



Time t-1

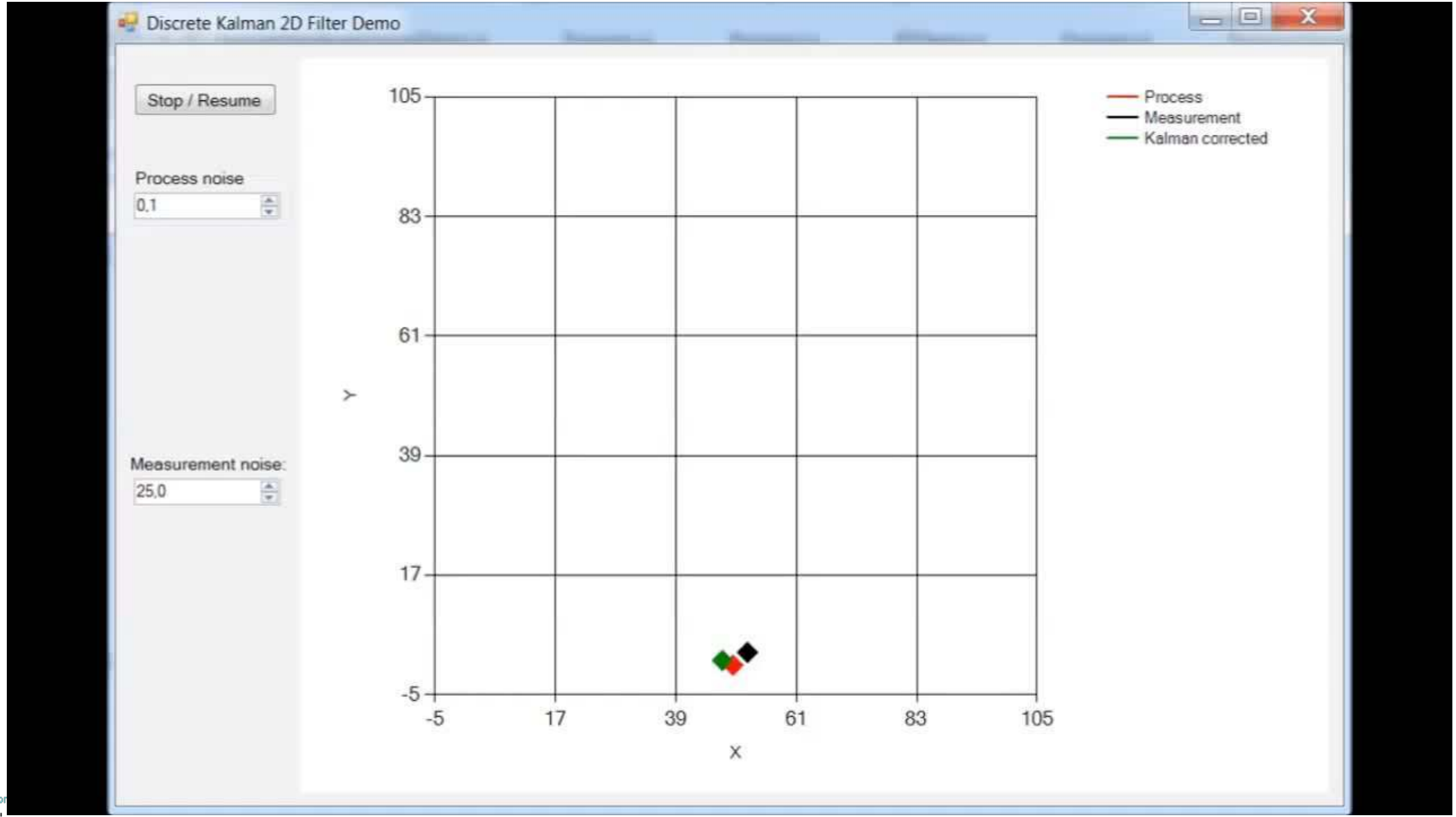
2. Measurement update or Correction

$$\begin{cases} \mu_t = \mu_t^- + \mathbf{G}_t(\mathbf{z}_t - \mathbf{C}\mu_t^-) \\ \Sigma_t = \Sigma_t^- - \mathbf{G}_t\mathbf{C}\Sigma_t^- \end{cases}$$

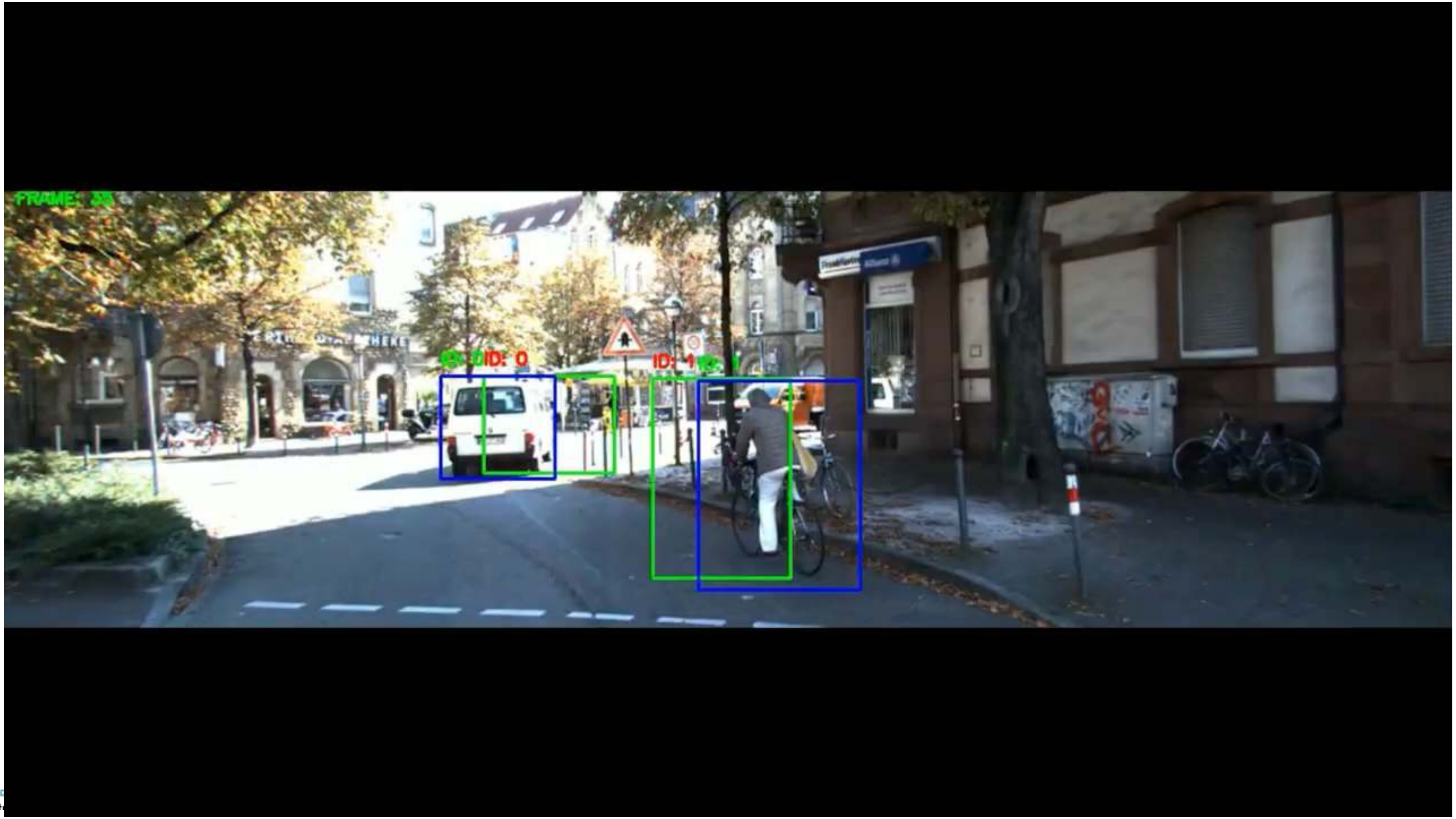
where $\mathbf{G}_t$ is the Kalman gain computed as:

$$\mathbf{G}_t = \Sigma_t^-\mathbf{C}^\top(\mathbf{C}\Sigma_t^-\mathbf{C} + \Sigma_\mathbf{v})^{-1}$$

Kalman filter

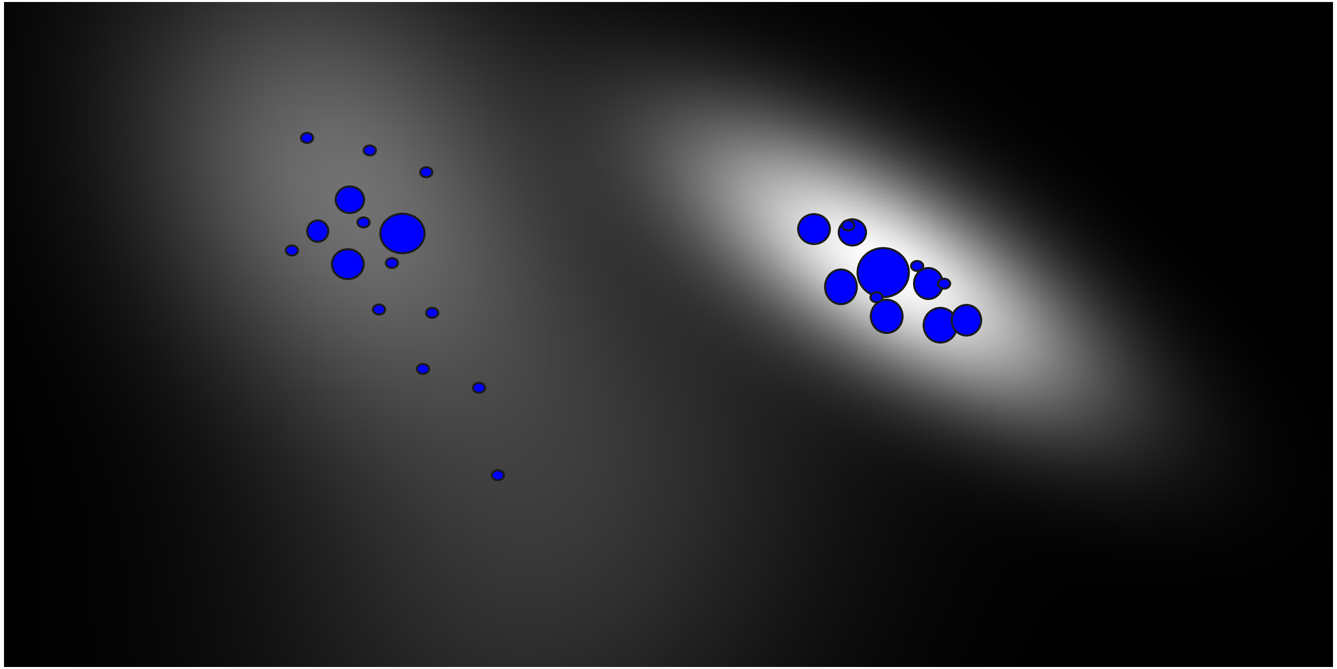


Kalman filter

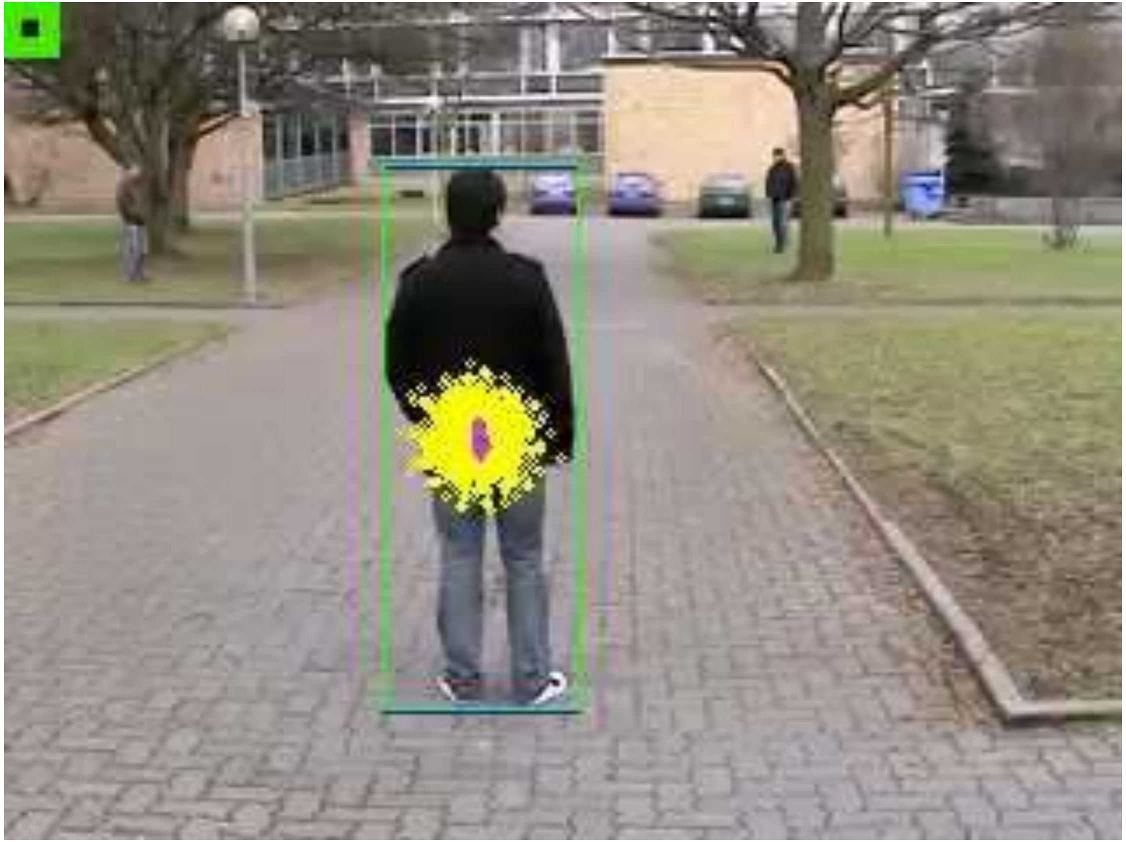


Particle Filter

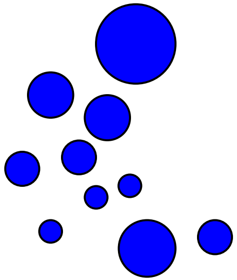
$$p(\mathbf{x}_t \mid \mathbf{Z}_t) \equiv \{(\mu_{t,i}, w_{t,i})\}_i$$



Particle Filter

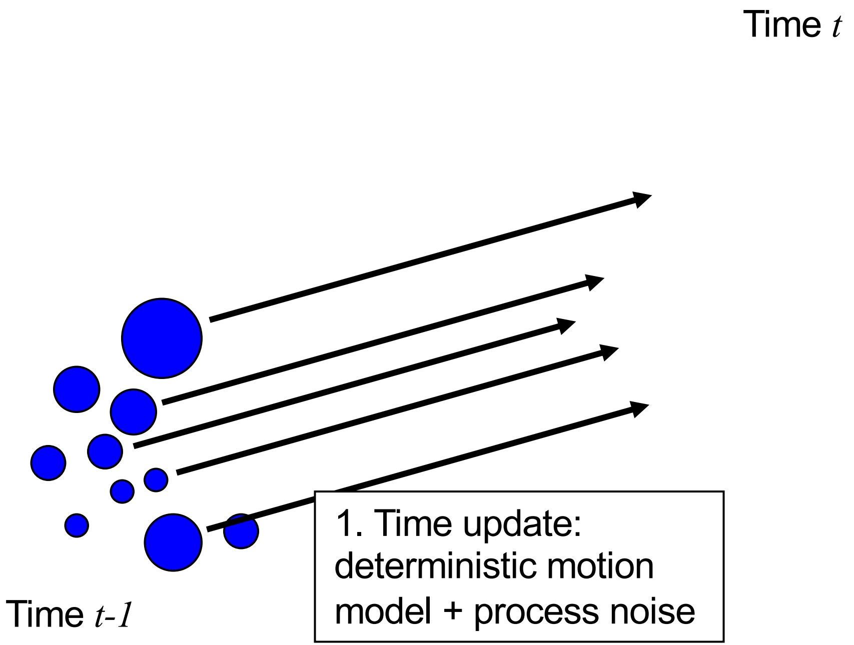


Particle filters

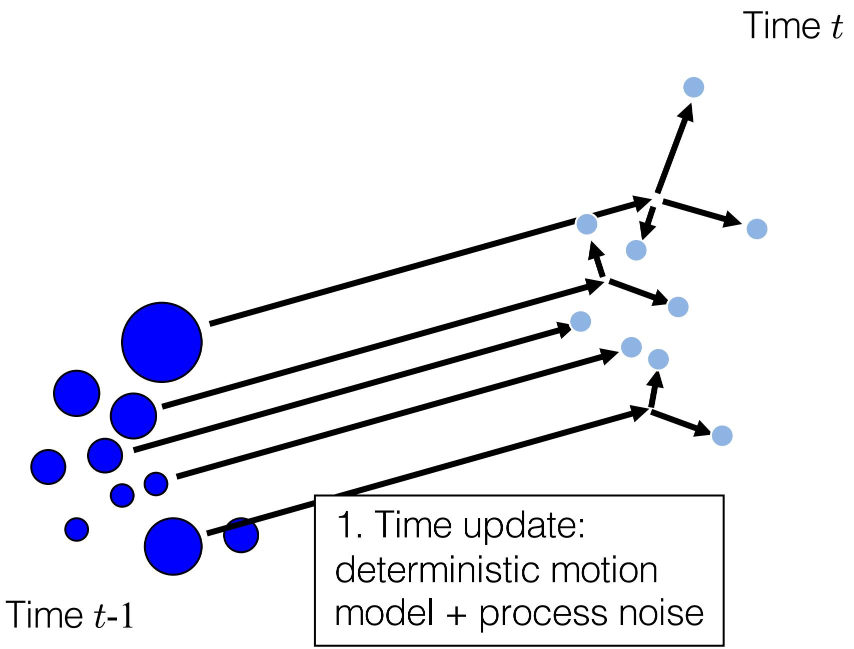


Time *t-1*

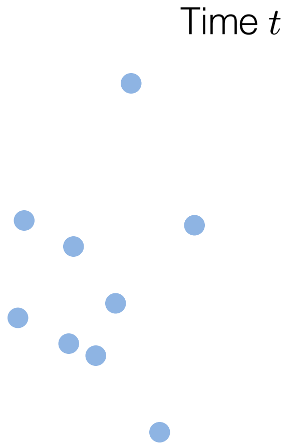
Particle filters



Particle filters

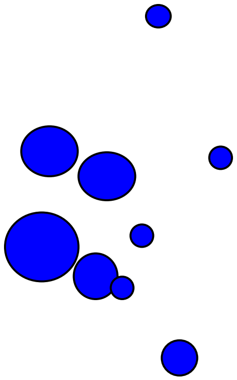


Particle filters



Particle filters

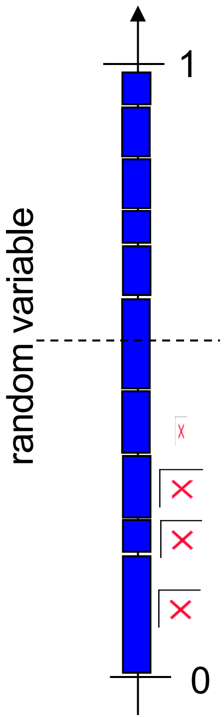
Time t

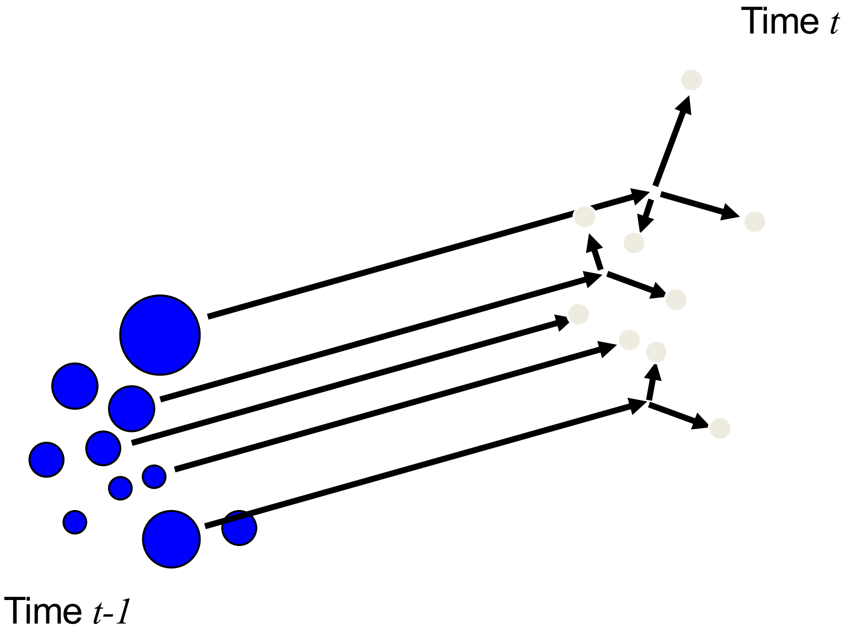


2. Measurement:
Weights are computed according to the likelihood:

Normalize the weights.

Efficient resampling of particles





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